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LETTER TO THE EDITOR

Quadratic Poisson brackets compatible with an algebra structure

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Abstract. Quadratic Poisson brackets on a vector space equipped with a bilinear multiplication are studied. A notion of a bracket compatible with the multiplication is introduced and an effective criterion of such compatibility is given. Among compatible brackets, a subclass of coboundary brackets is described, and such brackets are enumerated in a number of examples.

Investigation of many specific physical systems leads to a problem of quantization of Poisson brackets on spaces equipped with additional structures. Quantum algebras, or q-deformed algebras, have been useful in the investigation of many physical problems. The most studied case is when this structure is a Lie group structure. This leads to a celebrated quantum groups theory (see [1]). As a matter of fact, the research in q-groups was indeed originated from physical problems. The interest in q-groups and q-deformed algebras arose almost simultaneously in statistical mechanics as well as in conformal field theories, in solid-state physics as well as in the study of topologically non-trivial solutions of nonlinear equations.

As usual, Poisson bracket $\{\cdot,\cdot\}$ is understood as a Lie algebra structure on the space of smooth functions $C^\infty(M)$ satisfying the Leibnitz identity $\{fg,h\} = f\{g,h\} + \{f,h\}g$. If Jacobi identity is not required one speaks about pre-Poisson brackets. A mapping $F : M_1 \to M_2$ of two manifolds equipped with Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$, respectively, is called *Poisson* if

$$\{f \circ F, g \circ F\}_1 = \{f, g\}_2 \circ F.$$

In other words, a canonical mapping $F^* : C^\infty(M_2) \to C^\infty(M_1)$ is a Lie algebra homomorphism.

Let M be a smooth manifold, and $*$ be a multiplication, i.e. a mapping $M \times M \to M$. This immediately gives rise to a co-multiplication (diagonal) $\Delta : C(M) \to C(M) \otimes C(M)$ on the algebra $C(M)$ of functions on M (with the pointwise multiplication). Namely,

$$\Delta(f)(x, y) = f(x * y)$$

where $C(M \times M)$ is identified with $C(M) \otimes C(M)$. As it is well known, algebra $C(M)$ is responsible for the topological structure of M while a diagonal Δ reflects a multiplication structure.

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A Poisson bracket is said to be compatible with this multiplication of $*: M \times M \to M$ is a Poisson mapping where $M \times M$ is equipped with a product Poisson bracket. In other words, the following identity should be satisfied:

$$\Delta(\{f, g\}) = \{\Delta(f), \Delta(g)\} \tag{1}$$

where

$$\{p \otimes q, r \otimes s\} \stackrel{\text{def}}{=} \{p, r\} \otimes qs + pr \otimes \{q, s\}.$$

In particular, a bracket on a vector space compatible with addition structure is exactly a Berezin–Lie bracket, i.e. a linear bracket (speaking about linear, quadratic, etc, we will always mean that a bracket of two *linear* functions is linear, quadratic, etc, respectively).

The next-simplest case is quadratic Poisson brackets (see [2]). Such a bracket may be compatible only with a bilinear operation on the vector space, i.e. with an algebra structure. One of the most important cases, that of a full matrix algebra $\mathrm{Mat}(n, K)$ where $K = \mathbb{R}$ or $\mathbb{C}$, was investigated in detail in a series of works by Kupershmidt [3, 4, 5]. He also studied conditions under which a determinant (regarded as a function on $\mathrm{Mat}(n, K)$) is central.

In this letter a general description of Poisson brackets compatible with a given algebra structure is given. A case of quaternion algebra is given special consideration. All the compatible brackets are enumerated, and brackets for which a norm is central are explicitly described.

Quantization of Poisson brackets compatible with algebra structures, as well as their connection with quantum group theory is the subject of a forthcoming paper.

Consider a vector space A with a basis $\{e_i\}$, and let x^i be coordinate functions. A quadratic Poisson bracket is given by

$$\{x^i, x^j\} = c^{ij}_{kl} x^k x^l \tag{2}$$

a summation over repeated indices will always be assumed. Symbol c^{ij}_{kl} is skew-symmetric with respect to upper indices, but symmetry with respect to k and l is not generally assumed. Our task is to study brackets (2) compatible with the algebra structure in A given by the numerical structure constants a^i_{kl}, i.e.

$$e_k \cdot e_l = a^i_{kl} e_i. \tag{3}$$

With A being an algebra, a tensor square $A \otimes A$ can also be given an algebra structure by a component-wise multiplication. It is easy to see that the set of symmetric tensors $\mathrm{Symm}(A \otimes A) \subset A \otimes A$ is then a subalgebra, while the set of skew-symmetric tensors $\mathrm{Skew}(A \otimes A) \subset A \otimes A$ is a bimodule over $\mathrm{Symm}(A \otimes A)$. A linear mapping $D : B \to V$ from algebra B to a B-bimodule V is called a *differentiation* if it obeys the condition

$$D(p \cdot q) = pD(q) + D(p)q \tag{4}$$

for all $p, q \in B$.

Example 1. Let $r \in V$, and the algebra A is associative. Then the mapping

$$D_r(a) = ar - ra$$

is a differentiation.

Constants c^{ij}_{kl} from (2) also define an operator $C : \mathrm{Symm}(A \otimes A) \to \mathrm{Skew}(A \otimes A)$ by the formula

$$C(e_k \otimes e_l + e_l \otimes e_k) = (c^{ij}_{kl} + c^{ij}_{lk}) e_i \otimes e_j. \tag{5}$$

Theorem 1. The Poisson bracket (2) is compatible with the multiplication (3) if and only if the operator C given by (5) is a differentiation.

Proof. A co-multiplication Δ on the function algebra $C^\infty(A)$ is given by

$$\Delta(x^i) = a^i_{kl} x^k \otimes x^k.$$

The left-hand side of (1) equals $c^{ij}_{kl} a^k_{ef} a^l_{gh} x^e x^g \otimes x^f x^h$, so the coefficient at the term $x^m x^n \otimes x^t x^y$ is

$$c^{ij}_{kl}(a^k_{mt}a^l_{ny} + a^k_{nt}a^l_{my} + a^k_{my}a^l_{nt} + a^k_{ny}a^l_{mt}) \tag{6}$$

provided $m \neq n$, $t \neq y$, with obvious simplifications if some indices are the same.

The right-hand side of (1) equals $a^i_{pq} a^j_{rs}(x^p x^r \otimes c^{qs}_{wz} x^w x^z + c^{pr}_{uv} x^u x^v \otimes x^q x^s)$, so the coefficient at the term $x^m x^n \otimes x^t x^y$ is

$$(c^{pr}_{mn} + c^{pr}_{nm})(a^i_{pt}a^j_{ry} + a^i_{py}a^i_{rt}) + (c^{qs}_{ty} + c^{qs}_{yt})(a^i_{mq}a^j_{ns} + a^i_{nq}a^j_{ms}) \tag{7}$$

again, provided $m \neq n$, $t \neq y$. Compatibility condition means coincidence of (6) and (7) for all i, j, m, n, t, and y.

We now write differentiation identity (4) for the operator C given by (5) and $p = e_m \otimes e_n + e_n \otimes e_m$ and $q = e_t \otimes e_y + e_y \otimes e_t$. We suppose again that $m \neq n$, $t \neq y$, with the formulae being obviously simplified in case some equality holds. Then the coefficient at $e_k \otimes e_l$ in the left-hand side of (4) equals

$$c^{ij}_{kl}(a^k_{mt}a^l_{ny} + a^k_{nt}a^l_{my} + a^k_{my}a^l_{nt} + a^k_{ny}a^l_{mt}) \tag{8}$$

and in the right-hand side,

$$(c^{pr}_{mn} + c^{pr}_{nm})(a^i_{pt}a^j_{ry} + a^i_{py}a^j_{rt}) + (c^{qs}_{ty} + c^{qs}_{yt})(a^i_{mq}a^j_{ns} + a^i_{nq}a^j_{ms}). \tag{9}$$

Coincidence of (6) with (8), and of (7) with (9), is easily observed. □

Remark. All the above considerations do not make use of Jacobi identity, and therefore apply for general pre-Poisson brackets as well.

Example 2. [6] Consider a Poisson bracket

$$\{x^i, x^j\} = x^i x^j \qquad \text{for } i < j.$$

It is compatible with the co-multiplication

$$\Delta(x^i) = x^1 \otimes x^i$$

i.e. with the structure of the 'first column algebra' (algebra of $n \times n$-matrices having non-zero elements only in the first column). The corresponding differentiation is, however not given by a commutator described in example 1.

We now consider coboundary brackets. Hereafter all the algebras are assumed associative. Previously any quadratic pre-Poisson bracket compatible with the algebra A was identified with a differentiation $C : \mathrm{Symm}(A \otimes A) \to \mathrm{Skew}(A \otimes A)$. We now restrict our considerations to the *coboundary* case, when this differentiation is *internal*, i.e. given by

$$C(a) = ra - ar \tag{10}$$

where $r \in A \otimes A$. In some cases this, however, exhausts all the possible differentiations e.g. if $A = \mathrm{Mat}(n)$ is a full matrix algebra. For this algebra r is necessarily a skew-symmetric tensor.

An important function on $A = \mathrm{Mat}(n)$ is a determinant $\det(M)$. Kupershmidt [5] gives a necessary and sufficient condition on $r \in A \otimes A$ for $\det(M)$ to be a central function. It is

$$r^{ik}_{ij} = 0 \qquad \text{for all } j, k.$$

As a natural generalization of it, consider an algebra A and a function $\det(M)$, a determinant in its left regular representation.

Theorem 2. Let $r \in A \otimes A$ be as in (10), and $r_1, r_2 \in A$ be its traces (in a regular representation) in the first and the second component, respectively. Then det is a central function, if and only if for all $x \in A$ the element $y = xr_1 - r_2 x$ is a left annulator of A (i.e. $yz = 0$ for all $z \in A$).

The proof of theorem 2 simply copies Kupershmidt's computations in [5].

Consider coboundary pre-Poisson structures compatible with the algebra H of quaternions. A condition that a commutator (10) maps $\mathrm{Symm}(H \otimes H)$ into $\mathrm{Skew}(H \otimes H)$ gives that the element r can always be chosen skew-symmetric. Let $\|\cdot\|$ be a standard norm in H, and $x_1, \ldots, x_4$ be coordinate functions with respect to a standard basis $e_1 = \mathbf{1}$, $e_2 = \mathbf{i}$, $e_3 = \mathbf{j}$, $e_4 = \mathbf{k}$.

Theorem 3. All the coboundary pre-Poisson brackets compatible with a multiplication in H such that $\|\cdot\|$ is a central function are given by the following three-parameter family:

$$\{x^1, x^2\} = x^2(bx^3 + ax^4) - c((x^3)^2 + (x^4)^2) \tag{11}$$

$$\{x^1, x^3\} = x^3(ax^4 + cx^2) - b((x^4)^2 + (x^2)^2) \tag{12}$$

$$\{x^1, x^4\} = x^4(cx^2 + bx^3) - a((x^2)^2 + (x^3)^2) \tag{13}$$

$$\{x^2, x^3\} = -x^1(bx^2 + cx^3) \tag{14}$$

$$\{x^3, x^4\} = -x^1(ax^3 + bx^4) \tag{15}$$

$$\{x^4, x^2\} = -x^1(cx^4 + ax^2). \tag{16}$$

Moreover, all these brackets satisfy Jacobi identity (i.e. are not merely pre-Poisson, but genuine Poisson brackets).

Theorem 3 is proved by straightforward (however, cumbersome) computations.

Since the norm is a central function with respect to brackets (11)–(16), its level surfaces are Poisson submanifolds of H. A surface $\|x\| = 1$ is a group (inheriting its multiplication from H) isomorphic to $SU(2)$. Thus, theorem 3 gives us a three-parameter family of Poisson brackets in $SU(2)$ compatible with its Lie group structure, i.e., a three-parameter family of Poisson Lie groups.

It would be very interesting to study similar questions for the algebra $\mathbf{O}$ of octaves. Kupershmidt [5] studies connections between pre-Poisson brackets on an algebra A and on the group $\mathrm{Aut}(A)$ of its automorphisms. The point is that the connected component of $\mathrm{Aut}(\mathbf{O})$ is isomorphic to the exceptional compact simple Lie group G_2 which is currently a subject of intensive investigation.

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